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RIGHT CONGRUENCES FOR ω-REGULAR LANGUAGES

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1 Introduction

Let Σ be an alphabet and Σ^* be a free monoid generated by Σ. One of the main feature of the study of regular languages (of finite words) over Σ is the study of the right congruences (i.e., equivalence relations preserved under the concatenation from right) of Σ^*.

The next theorem is well known.

Theorem 1 (Myhill-Nerode) *The following three conditions for a language $L \subseteq \Sigma^*$ is equivalent.*
(1) *L is regular.*
(2) *L is a union of some equivalence classes of a right congruence of finite index.*
(3) *The equivalence relation $\sim$ defined by :*

$$u \sim v \text{ if for any } x \in \Sigma^* \; ux \in L \Leftrightarrow vx \in L$$

is a right congruence of finite index.

Moreover, for any regular language L, there exists a one-to-one correspondence between finite automata accepting L and right congruences of finite index recognizing L in the sense of (2) of the Myhill-Nerode's theorem.

In the case of ω-regular languages, the situation is not so simple as the case of regular languages. As shown in Example 1 below, there exists an ω-regular language L which does not have a unique deterministic minimum automaton accepting L. For the syntactic characterization of ω-regular languages, the results using (two-sided) congruence was obtained by Arnold [1], and the syntactic right congruence can recognize only the ω-languages in the subclass of ω-regular languages [4, 5].

Recently, Do Long Van, B.Le Saëc and I.Litovsky [3] give necessary and sufficient conditions for finite right congruences to recognize ω-regular languages. Maler and Staiger [4] introduce a notion of a family of right congruences, called a FORC, and show that an ω-language L is regular if and only if it is saturated by a finite FORC.

In this paper, we define simple and normal FORCs, and show that for any ω-language L, L is accepted by a deterministic Büchi (Muller, respectively) automaton if and only if it is covered (saturated) by a simple (normal) FORC. Moreover, there exists a one-to-one correspondence between simple (normal) FORCs covering (saturating) L and deterministic Büchi (Muller) automata accepting L.

2 Basic Definitions

For an alphabet Σ, we call a mapping $\alpha \in \Sigma^{\mathbf{N}}$ an *ω-word* over Σ, and write $\alpha = a_0a_1a_2\cdots$ where $a_n = \alpha(n)$ for each n. The set of all ω-words over Σ is denoted by Σ^ω, and that of all finite words over Σ is denoted by Σ^*, as usual. For $u = a_0a_1a_2\cdots a_m \in \Sigma^*$, we also denote the $(n+1)$th letter a_n of u as $u(n)$, $0 \le n \le m$.

The concatenation operation and prefix relation on Σ^* are generalized as follows. For $u \in \Sigma^*$ and $\alpha \in \Sigma^\omega$, $u\alpha$ is defined to be the ω-word obtained by concatenating u before α. If $\beta = u\alpha$, then we say that u is a *prefix* of β. For any $u \in \Sigma^*$ and $\alpha \in \Sigma^* \cup \Sigma^\omega$, we write $u \preceq \alpha$ if u is a prefix of α.

For $K \subseteq \Sigma^*$ and $L \subseteq \Sigma^\omega$, we define $KL = \{u\alpha \mid u \in K \text{ and } \alpha \in L\}$ and $K^\omega = \{v_1v_2\ldots \mid v_1, v_2, \ldots \in K - \{\epsilon\}\}$, where $v_1v_2\ldots$ is the ω-word obtained by concatenating $v_1, v_2, \ldots$ one after another.

For $\alpha \in \Sigma^* \cup \Sigma^\omega$, we define $\underline{\alpha} = \{a \mid a = \alpha(n) \text{ for some } n\}$, and $\underline{\underline{\alpha}} = \{a \mid a = \alpha(n) \text{ for infinitely many } n\}$. That is, $\underline{\alpha}$ is the set of letters appearing in α and $\underline{\underline{\alpha}}$ is the set of letters appearing infinitely many times in α.

A deterministic finite automaton over Σ is a quadruple $A = (Q, \Sigma, \delta, s)$, with the finite set Q of *states*, the *input alphabet* Σ, the *transition function* $\delta : Q \times \Sigma \to Q$, and the *initial state* $s \in Q$. (We do not include the usual set of accepting states in this definition.) For any $\alpha \in \Sigma^* \cup \Sigma^\omega$, the run $Run(A, \alpha)$ of A over α is the $\rho \in Q^* \cup Q^\omega$ such that $\rho(0) = s$ and $\rho(i+1) = \delta(\rho(i), \alpha(i))$ for any i.

For a deterministic finite automaton $A = (Q, \Sigma, \delta, s)$ and the set $F \subseteq Q$ of *accepting states*, the ω-language $I(A, F)$ accepted by (A, F) is defined by:

$$I(A, F) = \{\alpha \mid \underline{\underline{Run(A, \alpha)}} \cap F \neq \phi\}.$$

The automaton (A, F) is called a Büchi automaton.

For a deterministic finite automaton $A = (Q, \Sigma, \delta, s)$ and the set $\mathbf{F} \subseteq 2^Q$ of *accepting sets* of states, the ω-language $R(A, \mathbf{F})$ accepted by $(A, \mathbf{F})$ is defined by:

$$R(A, \mathbf{F}) = \{\alpha \mid \underline{\underline{Run(A, \alpha)}} \in \mathbf{F}\}.$$

The automaton $(A, \mathbf{F})$ is called a Muller automaton.

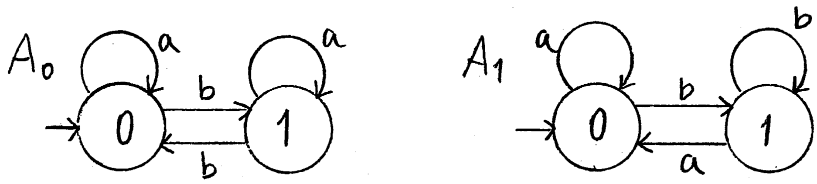
We define

$$\mathbf{I} = \{I(A, F) \mid (A, F) \text{ is a Büchi automaton over } \Sigma\},$$

$$\mathbf{R} = \{R(A, \mathbf{F}) \mid (A, \mathbf{F}) \text{ is a Muller automaton over } \Sigma\}.$$

That is, $\mathbf{I}$ ($\mathbf{R}$, respectively) is the class of ω-languages accepted by Büchi (Muller) automata over Σ. The class $\mathbf{R}$ is called the class of ω-regular languages over Σ. It is shown [2, 5, 6] that $\mathbf{I} \subset \mathbf{R}$, and $L \in \mathbf{R}$ if and only if $L = \bigcup_{i=1,n} J_i K_i^\omega$ for some regular languages $J_i, K_i \subseteq \Sigma^*$ $(i = 1, \ldots, n)$.

Example 1 Let $\Sigma = \{a, b\}$. The ω-language $(\Sigma^* b)^\omega$ is accepted by two essentially different two state Muller automata $(A_i = (\{0,1\}, \Sigma, \delta_i, 0), \mathbf{F}_i)$ $(i = 0, 1)$ with $\delta_0(p, a) = p$, $\delta_0(p, b) = 1-p$ for any $p = 0, 1$, $\mathbf{F}_0 = \{\{0, 1\}\}$, $\delta_1(p, a) = 0$, $\delta_1(p, b) = 1$ for any $p = 0, 1$ and $\mathbf{F}_1 = \{\{1\}, \{0, 1\}\}$.



Note that $(\Sigma^* b)^\omega$ is also accepted by the Büchi automaton $(A_1, \{1\})$.

A *right congruence* $\sim$ of Σ^* is an equivalence relation preserved under the concatenation from right, that is, $u \sim v$ implies $ux \sim vx$ for any $x \in \Sigma^*$. A right congruence is said to be *finite* if it has a finite number of equivalence classes.

Let $\sim$ be a right congruence of Σ^* and $u \in \Sigma^*$. The equivalence class of $\sim$ containing u is denoted by $[u]_\sim$, and we simply write $[u]$ if the right congruence $\sim$ is clear from the context.

For any finite right congruence $\sim$, we can assign a deterministic finite automaton $A_\sim = (\Sigma^*/\sim, \Sigma, \delta_\sim, [\epsilon])$, with $\delta_\sim([u], a) = [ua]$ for every $u \in \Sigma^*$ and $a \in \Sigma$. Conversely, for any deterministic finite automaton $A = (Q, \Sigma, \delta, s)$, we can assign a finite right congruence $\sim^A$ defined by: $u \sim^A v$ if and only if $\delta(s, u) = \delta(s, v)$. Note that these establish the one to one correspondence between finite automata and right congruences, i.e., $\sim^{A_\sim} = \sim$ and $A_{\sim^A}$ is isomorphic to A.

3 Simple FORCs and Normal FORCs

Recently, Maler and Staiger [4] defined a family of finite right congruences, called a FORC, to study the ω-languages. A FORC C is a family of finite right congruences $C = (\sim, \{\sim_{[u]} \mid u \in \Sigma^*\})$ such that for any $u, x, y \in \Sigma^*$, $x \sim_{[u]} y$ implies $ux \sim uy$. The right congruence $\sim$ is called the leading right congruence of C. We write $\sim_u$ for $\sim_{[u]}$ and $[x]_u$ for $[x]_{\sim_u}$. Thus $\sim_u = \sim_v$ and $[x]_u = [x]_v$ for any $x \in \Sigma^*$, if $u \sim v$.

We say that a FORC is *simple* if $x \sim_u y$ if and only if $ux \sim uy$ for any $u, x, y \in \Sigma^*$. In this case the FORC is determined by the leading right congruence $\sim$, so we call the FORC a simple FORC induced by $\sim$.

We say that a FORC is *normal* if $x \sim_u y$ if and only if
(1) $ux \sim uy$ and
(2) $\{[uv] \mid v \preceq x \text{ and } uv \sim uxz \text{ for some } z\}$
$= \{[uv] \mid v \preceq y \text{ and } uv \sim uyz \text{ for some } z\}$.
In this case the FORC is determined by the leading right congruence $\sim$, so we call the FORC a normal FORC induced by $\sim$.

An ω-language L is *covered* by a FORC $C = (\sim, \{\sim_{[u]} \mid u \in \Sigma^*\})$ if L is a finite union of ω-languages of the form $[u][v]_u^\omega$ with $u \sim uv$. An ω-language L is *saturated* by a FORC $C = (\sim, \{\sim_{[u]} \mid u \in \Sigma^*\})$ if $[u][v]_u^\omega \cap L \neq \phi$ implies $[u][v]_u^\omega \subseteq L$.

The following lemma proved in [4] assures that any FORC covers all of ω-words.

Lemma 2 ([4]) *For any FORC C over Σ, $\bigcup_{u \sim uv}[u][v]_u^\omega = \Sigma^\omega$.*

Lemma 3 *Any FORC C saturating L covers L.*

Proof. Let $K = \bigcup\{[u][v]_u^\omega \mid [u][v]_u^\omega \cap L \neq \phi\}$. $L \subseteq K$ is clear from the definition of K and Lemma 2. Since C saturates L, $K \subseteq L$. □

The converse of the above lemma does not always hold, as shown in the Example 2 below.

Example 2 *Let $\Sigma = \{a, b\}$ and $\sim$ be a right congruence with the equivalent classes $\{\epsilon \cup \Sigma^* a, \Sigma^* b\}$. Then the simple FORC induced by $\sim$ is $(\sim, \{\sim_\epsilon, \sim_b\})$, where $\sim_\epsilon = \sim$ and $\sim_b$ has the equivalent classes $\{\epsilon \cup \Sigma^* b, \Sigma^* a\}$. Thus, $(\Sigma^* b)^\omega = [b][\epsilon]_b$ is covered by C. Since $(\Sigma^* b)^\omega \cap [\epsilon][\epsilon]_\epsilon = (\Sigma^* b)^\omega \cap (\Sigma^* a)^\omega \neq \phi$ and $(\Sigma^* a)^\omega - (\Sigma^* b)^\omega \neq \phi$, $(\Sigma^* b)^\omega$ is not saturated by C.*

If a FORC C is normal, then it saturates any ω-languages covered by C.

Lemma 4 *If a FORC C is normal, C saturates L if and only if C covers L.*

Proof. It is enough to show that a normal FORC C saturating L covers L. Let $C = (\sim, \{\sim_u \mid u \in \Sigma^*\})$ be a normal FORC. For any u, v such that $u \sim uv$, we show that $\alpha \in [u][v]_u^\omega$ if and only if $\underline{\underline{Run(A_\sim, \alpha)}} = \{[uz] \mid z \preceq v\}$.

It is easy to see that if $\alpha \in [u][v]_u^\omega$ then $\underline{\underline{Run(A_\sim, \alpha)}} = \{[uz] \mid z \preceq v\}$. Assume $\underline{\underline{Run(A_\sim, \alpha)}} = \{[uz] \mid z \preceq v\}$. Then there exists $x, y_1, y_2, \cdots \in \Sigma^*$ such that $\alpha = xy_1y_2\ldots$, $u \sim x \sim xy_i$ for any i, and $\{[uz] \mid z \preceq v\} = \{[xz] \mid z \preceq y_i\}$ for any i. It means that $v \sim_u y_i$ for any i. Thus, $\alpha \in [u][v]_u^\omega$.

Now, assume $\alpha \in [u_1][v_1]_{u_1}^\omega \cap [u_2][v_2]_{u_2}^\omega$. It means that $\underline{\underline{Run(A_\sim, \alpha)}} = \{[u_1 x] \mid x \preceq v_1\} = \{[u_2 x] \mid x \preceq v_2\}$ and $[u_1][v_1]_{u_1}^\omega = [u_2][v_2]_{u_2}^\omega$. Hence C saturates any ω-languages covered by C. □

Lemma 5 *If L is covered by a simple FORC induced by $\sim$, then $L = \bigcup_{i=1}^n [u_i][\epsilon]_{u_i}$ for some $u_1, \ldots, u_n$.*

Proof. Let $(\sim, \{\sim_u \mid u \in \Sigma^*\})$ be a simple FORC induced by $\sim$. If $u \sim uv$, then $\epsilon \sim_u v$. Thus, $[u][v]_u = [u][\epsilon]_u$ for any $u \sim uv$. □

4 Main Results

Now we show the main results of this paper.

Theorem 6 *An ω-language L is in the class* **I** *if and only if it is covered by a simple FORC. Moreover, there exists a one-to-one correspondence between Büchi automata accepting L and simple FORCs covering L.*

Proof. Let $(A = (Q, \Sigma, \delta, s), F)$ be a Büchi automaton such that $L = I(A, F)$. Then

$$L = \bigcup_{q \in F} \{u \mid \delta(s, u) = q\}\{v \mid \delta(q, v) = q\}^{\omega}.$$

Consider the simple FORC induced by $\sim^A$. It is clear that $x \sim_u^A y$ if and only if $\delta(s, ux) = \delta(s, uy)$. Thus,

$$L = \bigcup_{\delta(s,u) \in F} [u][\epsilon]_u^{\omega}$$

Hence, L is covered by the simple FORC induced by $\sim^A$.

To show the converse, consider a simple FORC induced by $\sim$, and let $L = \bigcup_{i=1}^{n} [u_i][\epsilon]_{u_i}^{\omega}$. We define the Büchi automaton $(A_{\sim}, F)$ with $F = \{[u_i] \mid i = 1, \ldots, n\})$. Then an ω-word α is in L if and only if $\alpha \in [u_i][\epsilon]_{u_i}^{\omega}$ for some i if and only if α is accepted by $(A_{\sim}, F)$. Thus, $L = I(A_{\sim}, F)$. □

Theorem 7 *An ω-language L is in the class* **R** *if and only if it is saturated by a normal FORC. Moreover, there exists a one-to-one correspondence between Muller automata accepting L and normal FORCs saturating L.*

Proof. Let $(A = (Q, \Sigma, \delta, s), \mathbf{F})$ be a Muller automaton and $L = R(A, \mathbf{F})$. We define $run(q, a_1 \ldots a_n)$ is a finite sequence $q_0 \ldots q_n$ of states such that $q_0 = q$ and $q_i = \delta(q_{i-1}, a_i)$ for all i, $i = 1, \ldots, n$. Then

$$L = \bigcup_{q \in F \in \mathbf{F}} \{u \mid \delta(s, u) = q\}\{v \mid \delta(q, v) = q \text{ and } \underline{run(q, v)} = F\}^{\omega}$$

Consider the normal FORC induced by $\sim^A$. Then, for any u, x, y such that $ux \sim^A uy \sim^A u$, $x \sim_u^A y$ if $\underline{run(\delta(s, u), x)} = \underline{run(\delta(s, u), y)}$. It is easy to see that

$$L = \bigcup \{[u][v]_u^{\omega} \mid \delta(s, u) = \delta(s, uv) \text{ and } \underline{run(\delta(s, u), v)} \in \mathbf{F}\}$$

Hence, L is covered by the normal FORC induced by $\sim^A$. Since the FORC is normal, it saturates L.

To show the converse, consider the normal FORC induced by $\sim$, and let $L = \bigcup_{i=1}^{n}[u_i][v_i]_{u_i}^{\omega}$ with $u_iv_i \sim u_i$. We construct the Muller automaton $(A_\sim, \mathbf{F})$, where $\mathbf{F} = \{F_i \mid i = 1, \ldots, n\})$, and $F_i = \{[u_i z] \mid z \preceq v_i\}$ for any i. It is clear that $L \subseteq R(A, \mathbf{F})$.

Assume $\alpha \in R(A, \mathbf{F})$. Then $\underline{\underline{Run(A, \alpha)}} = F_i$ for some i. Since $[u_i] \in F_i$, α can be written as $\alpha = xy_1y_2\ldots$ so that $x \sim u_i$ and $y_j \sim_{u_i} v_i$ for all j. Thus, $\alpha \in L$. $\square$

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